

Day of the week



(a) Let k and n be positive integers, prove that for $k > 1$, when $(k-1)^{2n+1}$ is divided by k , the remainder is -1 .

(b) If today is Sunday, what day of the week is 2015^{2015} days after.

(a) Method 1

Let $P(n)$ be the proposition: $(k-1)^{2n+1} + 1 = a_n k$, where $a_n \in \mathbb{N}$, $k > 1$

We like to use Mathematical Induction to show that the proposition is true.

For $P(1)$,

$$(k-1)^3 + 1 = k^3 + 3k^2 + 3k = k(k^2 + 3k + 3) = a_1 k, \text{ where } a_1 \in \mathbb{N}$$

Assume that $P(m)$ is true for some $m \in \mathbb{N}$, that is

$$(k-1)^{2m+1} + 1 = a_m k \quad \dots (*)$$

For $P(m+1)$,

$$\begin{aligned} (k-1)^{2m+3} + 1 &= (k-1)^{2m+1}(k-1)^2 + 1 \\ &= (k-1)^{2m+1}(k^2 + 2k + 1) + 1 \\ &= (k-1)^{2m+1}(k^2 + 2k) + [(k-1)^{2m+1} + 1] \\ &= (k-1)^{2m+1}k(k+2) + a_m k, \text{ by } (*) \\ &= [(k-1)^{2m+1}(k+2) + a_m]k \\ &= a_{m+1}k, \text{ where } a_{m+1} = (k-1)^{2m+1}(k+2) + a_m \in \mathbb{N} \end{aligned}$$

$\therefore P(m+1)$ is true.

By the Principle of Mathematical Induction $P(n)$ is true $\forall n \in \mathbb{N}$.

Method 2

By Binomial theorem,

$$(k-1)^{2n+1} = k^{2n+1} - C_1^{2n+1}k^{2n} + C_2^{2n+1}k^{2n-1} - \dots + C_{2n}^{2n+1}k - 1$$

Hence,

$$\begin{aligned} (k-1)^{2n+1} + 1 &= k^{2n+1} - C_1^{2n+1}k^{2n} + C_2^{2n+1}k^{2n-1} - \dots + C_{2n}^{2n+1}k \\ &= k(k^{2n} - C_1^{2n+1}k^{2n-1} + C_2^{2n+1}k^{2n-2} - \dots + C_{2n}^{2n+1}) \end{aligned}$$

Since C_r^{2n+1} are positive integers (you can prove this by mathematical induction if you want)

$(k-1)^{2n+1} + 1$ is divisible by k .

Therefore, for $k > 1$, when $(k-1)^{2n+1}$ is divided by k , the remainder is -1 .

(b) Take $k = 2016$, $n = 1007$ in (a),

$$(2016-1)^{2(1007)+1} = 2015^{2015} \text{ when divided by } 2016 = 7 \times 288$$

the remainder is -1 .

If today is Sunday, one day before will be Saturday.

Epilogue

By the time 2015^{2015} days, our sun has exhausted its nuclear fuel and collapsed to form a white dwarf star. There will not be any solar system. Also, astronomers have been saying that our home galaxy, the Milky Way, is destined 4.5 billion years later to collide with the next-nearest spiral galaxy, in the direction of the constellation Andromeda. When they collide, our sun will likely be flung into a new region of galactic space. For pessimists, there will not be any human beings. For optimists, humans will learn inter-galactic travel. The definition of time and space will be very different from today. By that time, there may not be any Sunday or Saturday. I cannot wait for that day to come.

Yue Kwok Choy

10/7/2015